On a curious property of 3435.

Daan van Berkel

November 18, 2009

Abstract

Folklore tells us that there are no uninteresting natural numbers. But some natural numbers are more interesting then others. In this article we will explain why 3435 is one of the more interesting natural numbers around.

We will show that 3435 is a *Munchausen number* in base 10, and we will explain what we mean by that. We will further show that for every base there are finitely many Munchausen numbers in that base.

Folklore tells us that there are no uninteresting natural numbers. The argument hinges on the following observation: *Every subset of the natural numbers is either empty, or has a smallest element.*

The argument usually goes something like this. If there would be any uninteresting natural numbers, the set $\mathcal{U}$ of all these uninteresting natural numbers would have a smallest element, say $u \in \mathcal{U}$. But u in it self has a very remarkable property. u is the smallest uninteresting natural number, which is very interesting indeed. So $\mathcal{U}$, the set of all the uninteresting natural numbers, can not have a smallest element, therefore $\mathcal{U}$ must be empty. In other words, all natural numbers are interesting.

Having established this result, exhibiting an interesting property of a specific natural number is often left as an excercise for the reader. Take for example the integer 3435. At first it does not seem that remarkable, until one stumbles upon the following identity.

$$3435 = 3^3 + 4^4 + 3^3 + 5^5$$

This coincidence is even more remarkable when one discovers that there is only one other natural number which shares this property with 3435, namely $1 = 1^1$.

In this article we will establish the claim made and generalize the result.

Munchausen Number

Through out the article we will use the following notation. $b \in \mathbb{N}$ will denotate a base and therefore the inequility $b \geq 2$ will hold throughout the article. For every natural number $n \in \mathbb{N}$, the *base b representation of n* will be denoted by $[c_{m-1}, c_{m-2}, \ldots, c_0]_b$, so $0 \leq c_i < b$ for all $i \in \{0, 1, \ldots, m-1\}$ and $n = \sum_{i=0}^{m-1} c_i b^i$. Furtheremore, we define a function $\theta_b : \mathbb{N} \to \mathbb{N} : n \mapsto \sum_{i=0}^{m-1} c_i^{c_i}$, where $n = [c_{m-1}, c_{m-2}, \ldots, c_0]_b$. We will further adopt the convention that $0^0 = 1$, in accordance with $1^0 = 1$, $2^0 = 1$ etcetera.

Definition An integer $n \in \mathbb{N}$ is called a *Munchausen number in base b* if and only if $n = \theta_b(n)$. ∘

So by the equality in the introduction we know that 3435 is a Munchausen number in base 10.

Remark A related concept to Munchausen number is that of Narcissistic number. (See for example [1], [2] and [3].)

The reason for picking the name Munchausen number stems from the visual of raising oneself, a feat demonstrated by the famous Baron von Munchausen ([4]). Andrew Baxter remarked that the Baron is a narcissistic man indeed, so I think the name is aptly chosen. ◁

The following two lemmas will be used to proof the main result of this article: for every base $b \in \mathbb{N}$ there are only finitely many Munchausen numbers in base b.

Lemma 1 *For all* $n \in \mathbb{N}$*:* $\theta_b(n) \leq (\log_b(n) + 1)(b-1)^{b-1}$. ◇

Proof Notice that the function $x \mapsto x^x$ is strictly increasing if $x \geq \frac{1}{e}$. This can be seen from the derivative of x^x which is $x^x(\log(x) + 1)$. This last expression is clearly positive for $x > \frac{1}{e}$. Together with the definition of $0^0 = 1$, we see that x^x is increasing for all the nonnegative integers.

For all $n \in \mathbb{N}$ with $n = [c_{m-1}, c_{m-2}, \ldots, c_0]_b$ we have the ineqalities $0 \leq c_i \leq b - 1$ for all i within $0 \leq i < m$.
So $\theta_b(n) = \sum_{i=0}^{m-1} c_i^{c_i} \leq \sum_{i=0}^{m-1} (b-1)^{b-1} = m \times (b-1)^{b-1}$.

Now, the number of digits in the base b represantation of n equals $\lfloor \log_b(n) + 1 \rfloor$. In other words $m := \lfloor \log_b(n) + 1 \rfloor \leq \log_b(n) + 1$.

So $\theta_b(n) \leq (\log_b(n) + 1)(b-1)^{b-1}$ □

Lemma 2 *If* $n \in \mathbb{N}$ *and* $n > 2b^b$ *then* $\frac{n}{\log_b(n)+1} > (b-1)^{b-1}$. ◇

Proof Let $n \in \mathbb{N}$ such that $n > 2b^b$. Notice that $x \mapsto \frac{x}{\log_b(x)}$ is strictly increasing if $x > e$. To see this notice that the derivative of $\frac{x}{\log_b x}$ is $\log(b)\frac{\log(x)-1}{\log^2(x)}$ which is positive for $x > e$. Furthermore $\log_b(2) + 1 \leq 2 \leq b = b\log_b(b)$.

Now, because $n > 2b^b > e$, from the following chain of ineqalities:

$$\frac{n}{\log_b(n) + 1} > \frac{2b^b}{b\log_b(b) + \log_b(2) + 1} \geq \frac{2b^b}{2b\log_b(b)} = b^{b-1} > (b-1)^{b-1}$$

we can deduce that $\frac{n}{\log_b(n)+1} > (b-1)^{b-1}$ □

With both lemma's in place we can present without further ado the main result of this article.

Proposition 3 *For every base $b \in \mathbb{N}$ with $b \geq 2$: there are only finitely many Munchausen numbers in base b.* ◇

Proof By the preceding lemma's we have, for all $n \in \mathbb{N}$ with $n > 2b^b$: $n > (\log_b(n) + 1)(b-1)^{b-1} \geq \theta_b(n)$.

So, in order for n to equal $\theta_b(n)$, n must be less then or equal to $2b^b$. This proves that there are only finitely many Munchausen numbers in base b. □

Exhaustive Search

The proposition in the preceding section tells use that for every base $b \in \mathbb{N}$, Munchausen numbers in that base only occur within the interval $[1, 2b^b]$. This makes it possible to exhaustively search for Munchausen numbers in each base.

Figure 1 lists all the Munchausen numbers in the bases 2 through 10. So for example in base 4, 29 and 55 are the only non-trivial Munchausen numbers. Furthermore, the base 4 representation of 29 and 55 have a striking resemblance. For $29 = [1,3,1]_4 = 1^1+3^3+1^1$ and $55 = [3,1,3]_4 = 3^3+1^1+3^3$.

The sequence of Munchausen numbers is listed as sequence A166623 at the OEIS. (See [5]. For the related sequence of Narcissistic numbers see [6])

The code in listing 1 is used to produce the numbers in figure 1. There are two utility functions. These are munchausen and next. munchausen calculates $\theta_b(n)$ given a base b representation of n. next returns the base b representation of $n+1$ given a base b representation of n.

I would like to conclude this article with a question my wife asked me while I was writing this: "But what about 20082009?"

Figure 1: Munchausen numbers in base 2 through 10.

Base	Munchausen Numbers	Representation
2	1, 2	$[1]_2$, $[1,0]_2$
3	1, 5, 8	$[1]_3$, $[1,2]_3$, $[2,2]_3$
4	1, 29, 55	$[1]_4$, $[1,3,1]_4$, $[3,1,3]_4$
5	1	$[1]_5$
6	1, 3164, 3416	$[1]_6$, $[2,2,3,5,2]_6$, $[2,3,4,5,2]_6$
7	1, 3665	$[1]_7$, $[1,3,4,5,4]_7$
8	1	$[1]_8$
9	1, 28, 96446, 923362	$[1]_9$, $[3,1]_9$, $[1,5,6,2,6,2]_9$, $[1,6,5,6,5,4,7]_9$
10	1, 3435	$[1]_{10}$, $[3,4,3,5]_{10}$

Listing 1: GAP code finding Munchausen numbers

```
next := function(coefficients, b)
        local i;
        coefficients[1] := coefficients[1] + 1;
        i := 1;
        while coefficients[i] = b do
                coefficients[i] := 0;
                i := i + 1;
                if (i <= Length(coefficients)) then
                        coefficients[i] := coefficients[i] + 1;
                else
                        Add(coefficients, 1);
                fi;
        od;
        return coefficients;
end;

munchausen := function(coefficients)
        local sum, coefficient;
        sum := 0;
        for coefficient in coefficients do
                sum := sum + coefficient^coefficient;
        od;
        return sum;
end;

for b in [2..10] do
        max := 2*b^b;
        n := 1; coefficients := [1];
        while n <= max do
                sum := munchausen(coefficients);

                if (n = sum) then
                        Print(n, "\n");
                fi;

                n := n + 1;
                coefficients := next(coefficients, b);
        od;
od;
```

References

[1] Clifford A. Pickover. *Wonders of Numbers*. Oxford University Press, 2001.

[2] Wikipedia. Narcissistic Number. http://en.wikipedia.org/wiki/Narcissistic_number.

[3] Wolfram Math World. Narcissistic Number. http://mathworld.wolfram.com/NarcissisticNumber.html.

[4] Wikipedia. Baron Munchhausen. http://en.wikipedia.org/wiki/Baron_Munchhausen.

[5] The On-Line Encyclopedia of Integer Sequences. A166623. http://www.research.att.com/~njas/sequences/A166623.

[6] The On-Line Encyclopedia of Integer Sequences. A005188. http://www.research.att.com/~njas/sequences/A005188.