What is the difference between remembering someone posting a letter and remembering the square root of 2?

Gary E. Davis

School of Education, University of Southampton, United Kingdom

I claim that what distinguishes the mathematical thought of people capable of easy mathematical attainment is a heightened use of episodic memory about mathematical objects. To the contrary, people who find mathematics hard and sterile probably do so because their memory of it is semantic in the most limited sense.

Learning mathematics ought to be easy, but it's not. Why is that?

In this paper I want to explore the possibility that, as well as the constrictions of rigour, one reason mathematics is so hard for so many people is that mathematicians do not think about mathematics the way that other people do. In order to make this a non-trivial statment, let me re-phrase it differently: there are ways of thinking about mathematics that make it far easier than other ways, and most people adopt the harder way. Moreover, by the very nature of their habits of thought - their habitual way of constructing their world - they are highly unlikely to ever see that there is an easier way. The idea that some people view mathematics in a way that makes it easier for them to learn is not new: it is certainly explicit in Gray and Tall (1994).

What I want to say is based on an ancient distinction between episodic and semantic memory. This distinction, and its psychological ramifications, has been made much clearer in recent times through the work of Endel Tulving (Tulving, 1983). Tulving's early view of the distinction between episodic and semantic memory is summed up by him as follows:

"Episodic memory is concerned with unique, concrete, personal experiences dated in the rememberer's past; semantic memory refers to a person's abstract, timeless knowledge of the world that he shares with others. Distinctions of this kind had been quite familiar to philosophers interested in problems of memory, but their implications for the psychological study of memory had not been explored. ..... Episodic memory, I suggested is a system that receives and stores information about temporally dated episodes or events, and temporal-spatial relations among them. .... Semantic memory, I suggested, 'is the memory necessary for the use of language. It is a mental thesaurus, organized knowledge a person possesses about words and other verbal symbols, their meanings and referents, about relations among them, and about rules, formulas and algorithms for the manipulation of the symbols, concepts and relations'." (Tulving, 1983, pp.v, 21)

On the face of it, semantic memory sounds just like the sort of memory that is applicable to learning mathematics. Indeed, without such a memory system that was highly developed it is difficult to imagine how we could ever learn mathematics. Epsiodic memory, on the other hand, is what gives our individual lives continuity, connectedness and a sense of reality. Does this mean that mathematics is forever assigned to a part of our memories that does not deal with such matters, and must always therefore be seen as a "timeless other", existing somewhere out there as a system of codified rules and procedures? The answer is of course "no". One only has to listen to a creative mathematician talk about their work to realise that this is not how they see mathematics. Yet for most people this *is* their view of mathematics: a rule-driven, bloodless, passionless, activity, situated nowhere in time or space.

How can it be that mathematicians have such a different view of their activities from the general populace's view? One answer, I believe, is that people who learn mathematics relatively easily, who enjoy it with a passion, are in fact re-membering it through episodic memory as well as through their semantic memory system. For such people mathematics does not have to made meaningful or "real-world": it is already alive because it is situated in time and space in their lived experience in the world.

Hiebert and Lefevre (1986) use Tulving's episodic/semantic distinction in the context of conceptual and procedural knowledge in mathematics. They say, rightly in my opinion, that "The distinction between conceptual and procedural knowledge that we elaborate ... is not synonymous with any of these distinctions, but it draws upon all of them." (p. 1). The episodic/semantic distinction is therefore not to be equated simply with a conceptual/procedural distinction, although there are parallels. It is, for example, entirely possible to recall semantically whilst thinking conceptually. Indeed Tulvings explanation of semantic memory certainly involves relations between symbols and their referents.

Tulving (1983) states quite clearly that: "the absence of phenomenal experience, or concepts corresponding to it, characterizes just about every conceptual account of memory that we have had in experimental psychology, although paradoxically it is particularly true of cognitive theories of memory." (p. 125)

In the psychological study of the learning of mathematics this is a factor that seems to me to play a pivotal role: the phenomenal experience of the student engaging in mathematical activity. Let me give an example that I hope will elucidate my main point.

I asked several mathematics staff and graduate students from La Trobe University what comes to mind when they think of eigenvalues. The responses I got were typically as follows:

* Contraction maps or expanding maps.

* Straight lines being stretched; the Greek letter λ; Jordan decomposition; some sort of operation on some sort of space.

* the German language; $(n\pi)^2$ (the eigenvalues of the differential operator $Ly = y''$); the symbols λ, μ, Λ

*I imagine some sense of the "size" of a matrix related to the determinant.

* Why are we doing this? (*memory of lectures as an undergraduate*). Then reading Halmos' book and finding he called them something else.

* Characteristic numbers, Chern-Weil characteristic classes, cobordism, Smale and Milnor; Jordan canonical form, change of field of scalars, matrices with entries in non-commutative rings: it is ad-bc, not da-cb; Anosov, exp,

hyperbolicity, chaos.

* I think of numbers representing the simple "rescaling" of an object after some operation has been performed on that object.

Contrast these responses with those of a number of third year mathematics students in the same university. These students were listening to a fellow student explain his reasoning about symmetries of a cube. At one point he introduced eigenvalues and several students stopped him, apparently puzzled, asking what eigenvalues had to do with the problem. After questioning by me the entire class admitted that, for them, eigenvalues weren't *about* anything - one simply calculated them. Their memory of eigenvalues - what they re-call - is indeed semantic in the narrowest sense. There was nothing, apparently, in their memory that situated eigenvalues in time and place with a sense of meaning. They may well have been able to remember that it was a hot spring morning in a particular lecture room when they first encountered the notion explicitly, but that is a memory of themselves, of their situatedness, not episodic memory of eigenvalues in the sense that the La Trobe mathematicians exhibited it.

How is it then that some people have apparent episodic memory for what appear to be semantic events? How is it that the highly algorithmic, abstract, timeless world of mathematics is for them alive, highly connected, with a sense of continuity and situatedness in both time and space?

At first glance it seems that any one person would have to use episodic memory in much the same way as anyone else: episodic memory relates to where and when, whereas semantic memory relates to how, why, and in relation to what. However, there is another factor that is important in the distinction betwen episodic and semantic memeory, and it is the notion of context. Episodic memory is highly context dependent. In some sense this is to what the "when" and "where" of episodic memeory refers. To the contrary, semantic memory is highly context independent. It is "knowledge about" something. That the earth attracts massive objects to it, is (to the best of our understanding) not dependent upon when and where those objects are. What I suspect is that to a person who learns mathematics easily and becomes good at it, the mathematical "knowledge" so obtained retains within it elements of context. The re-call, for example, of a linear map, is for such people, not one of a definition such as $L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$ (although the ability to call to mind such semantic knowledge is essential), rather it is something like a mental picture of a reflection in the plane or in space, or a rotation about a fixed axis in space, or something of that nature. These re-membered mathematical objects appear contextually, as if they were genuine objects: in that sense their re-call has an aspect of episodic memory to it. This is in contrast to what Furlong (1951) says about the difference between remembering seeing someone post a letter and remembering the square root of a number:

"In the former case the mind looks back to a past event: we recollect, reminisce, retrospect; there is imagery. In the latter case this looking back is absent, and there is little or no imagery. We have retained a piece of information; that is all. There is retentiveness but not retrospection." (p. 6)

This characterizes for me the essential difference between a mathematical and non-mathematical bent of mind. A mathematical mind has a rich, and continually increasing, set of images associated with, for instance, the square root of a number. These images might include a right triangle, a ruler and compass construction, a rapidly converging sequence of rational numbers obtained from Newton's algorithm, A Dedekind section of the rational numbers, a construction of an algebraic number field from an irreducible polynomial, a continued fraction, a Conway game, and more. It is an imageless, decontextualized notion of something as basic as a square root that prevents many people from learning mathematics. It is as if they have decided to cut their own lively, creative mathematical actions out of their memory. But this is a mistaken notion because a person who re-members a square root in such an imageless way surely never had a lively creative action that related to square roots! Such students cannot remember relatively episodically because the only thing for them to re-member - to build again in their minds - is the dis-embodied, decontextualized written sign for a square root.

Once again I asked the La Trobe University staff and graduate students to tell me what comes to mind when they think of the square root of 2. A selection of their answers below shows the rich diverstiy of contextualized memory, including significant examples of imagery:

* $\sqrt{2}$; 1.414; Pythagoras; Samos in Greece.

* 1.414; right triangle of short side lengths 1.

* I imagine a square whose area is 2 such that each edge has length $\sqrt{2}$.

* The proof that it's irrational: fractions in lowest forms; polynomials; geometry.

* I think of a number between 1 and 2 with an infinite non-repeating decimal representation. The latter I tend to think of in terms of rational approximants to the number.

* I think of $\sqrt{2}$ as the number x such that $x^2 = 2$ and also that 1.4142 is a rational approximation. The right-angled triangle 1,1,$\sqrt{2}$ also comes to mind.

* Right triangles, scalar product, Riemannian metrics, curved spaces; $Q(\sqrt{2})$ is finite-dimensional: life is easy here.

The episodic/semantic distinction in memeory orients us to an individual's actions and their recollection by that individual. Indeed Tulving (1989) quotes Clarapede (1911) as saying that there are two sorts of memory: that relating to representations, and that relating to representations and the self. This is a central issue for mathematics learning: the objects and procedures of mathematics are not remembered in a purely semantic way by good mathematicians - they are remembered in relation to the self. And this memory in relation to the self is a memory of the self's *actions*. So, it would seem to be easy to allow anyone to be good at mathematics: promote the constructions of individuals and the re-collection of those constructions, as people do in constructivist-oriented classrooms (Martin, Pateman and Higa, 1993; Martin and Pateman, 1993; Pateman and Higa, 1993; Steffe and D'Ambrosio, 1995).

However, most of us have been through fairly dry classes in mathematics. Why do some of us come out with a passion for the subject, and and a relative ease in learning it, whilst others do not? We may ask: why do some people actively seek to experiment in mathematics, and why are they so easily able to episodically recall those experiments? I remember once telling my colleague Andrew Waywood - then my student - that one could get a better feeling for the shape of a function, for its derivative, if one imagined placing oneself on the function and sliding along it. This gives me a heightened feeling for why exponential functions are so different to monomials, and just how peculiar is the toplogist's sine curve - the graph of $sin(\frac{1}{x})$. A former colleague of mine, Ken Miles, could solve complicated topological problems in his head: he confessed to having continually moving images, even when he was thinking of algebraic formulas which he would see weaving and dancing. A grade 3 student whom Robert Hunting and I worked with several years ago showed, after a short time, exceptional ability in rational number problems. He confessed to us that he habitually split numbers into their parts - found their divisors, in other words. This mental activity, constantly practised and recalled, allowed him to easily evaluate complicated rational number comparisons: such as whether $\frac{3}{5}$ was bigger than $\frac{5}{8}$ and by how much. So now let me ask: does the process of putting life into mathematics, of being able to re-call mathematics in a contextualized wat, with episodic features, have to do with purposeful, intentional activity in mathematical settings? The answer, I believe, will turn out to be "yes". How we might find this out is by doing PET and MRI studies of people thinking about mathematics. If intentional brain centres can be isolated by these techniques there is also a possibility that we can distinguish episodic and semantic areas, and so relate different levels of mathematical achievement to differing modes of thought.

For the reasons I have outlined, I am pessimistic about a majority of people ever being able to learn mathematics effectively. It is not that most people *could* *not* understand mathematics relatively easily. The problem is, I believe, that in order to recall mathematics so as to be able to do it easily they have to recall it episodically. But in order to do that there must be episodes and contexts to recall. These episodes and contexts require active construction by individuals. Why, however, should a young student's mind actively engage with mathematics when they don't yet know what it is about? It is not, after all, an everyday subject that most people talk about as they do the weather, their health, or the activities of their friends and neighbours. But perhaps it could be: maybe it could be as lively a reminiscence in one's mind remembering the square root of 2 as remembering someone post a letter.

References

Clarapede, E. (1991) *Arch. Psychol.* 11, 79-90.

Furlong, E.J. (1951) *A study in memory.* London: Nelson.

Gray, E.M. & Tall, D.O. (1994) Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. J. Res. Math. Ed., 25 (2), 116-140.

Hibert, J. & Lefevre, P. (1986) Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (ed.) *Conceptual and procedural knowledge: The case of mathematics.* Hillsdale: Lawrence Erlbaum.

Martin, W.G., Pateman, N.A. & Higa, J. (1993) Developing a constructivist learning environment in high school geometry. N.C.T.M., Seattle.

Pateman, N.A. & Higa, J. (1993) A model for curriculum reform: Problem-based geometry in a heterogeneous classroom. AERA, Atlanta, Georgia.

Martin, W.G. & Pateman, N.A. (1993) Problem-based geometry: implementing changes in content and methodology. AERA, Atlanta, Georgia.

Steffe, L.P. & D'Ambrosio, B.S. (1995) Toward a working model of constructivist teaching - a reaction to Simon. *J. Res. Math. Ed.*, 26 (2), 146-159.

Tulving, E. (1983) *Elements of episodic memory.* Oxford: Oxford University Press.